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MEMORANDUM

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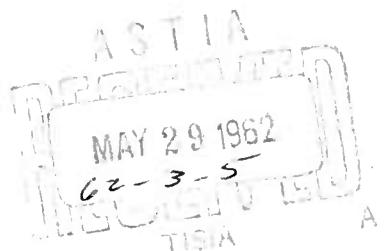
MAY 1962

ORBIT DETERMINATION AS  
A MULTIPOINT BOUNDARY VALUE PROBLEM  
AND QUASILINEARIZATION

Richard Bellman, Harriet Kagiwada and Robert Kalaba

PREPARED FOR:

UNITED STATES AIR FORCE PROJECT RAND



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PREFACE

This is one of a continuing series of studies aimed at more fully exploiting the capabilities of modern digital computers for the solution of scientific and technological problems of interest to the Air Force.

In this Memorandum a new approach to the problem of the determination of orbits of bodies is described and some numerical evidence showing its efficiency is adduced.

SUMMARY

Many problems in mechanics and applied mathematics in general require the solution of multipoint nonlinear boundary value problems. A computational approach to their solution, based on the idea of quasilinearization, is described and the results of an application to the computational determination of orbits are presented.

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ORBIT DETERMINATION AS A MULTI-POINT BOUNDARY VALUE PROBLEM  
AND QUASILINEARIZATION

1. INTRODUCTION

The basic ideas underlying the solution of boundary value problems via quasilinearization are discussed in [1] and [2]. Results of some numerical experiments involving two-point boundary value problems for nonlinear ordinary differential equations (Euler equations) are available in [2] and [3], while results for nonlinear partial differential equations are contained in [4]. The purpose of this note is to describe an application to the computational solution of some general  $N$ -point boundary value problems and to present an application to orbit determination.

2. AN  $N$ -POINT BOUNDARY VALUE PROBLEM

Consider an  $N$ -dimensional vector  $x(t) = (x_1(t), x_2(t), \dots, x_N(t))$  which is a solution of the vector system of equations

$$(2.1) \quad \dot{x} = f(x, t), \quad 0 \leq t \leq b,$$

and in addition satisfies the  $N$  conditions

$$(2.2) \quad \sum_{j=1}^N a_j(t_i) x_j(t_i) = b_i, \quad i = 1, 2, \dots, N,$$

where

$$(1.3) \quad 0 \leq t_1 \leq t_2 \leq \dots \leq t_N \leq b.$$

Let us assume that the equations (2.1) and (2.2) possess a

unique solution on the interval  $[0, b]$ . Our objective is to provide an efficient computational algorithm for determining the function  $x(t)$  on the interval  $[0, b]$ .

### 3. SOLUTION VIA QUASILINEARIZATION

Let  $x^{(0)}(t)$  be an initial approximation on the given interval. The  $(k + 1)$ -st approximation,  $x^{(k+1)}(t)$ , is to be obtained from the  $k$ -th approximation,  $x^{(k)}(t)$ , through use of the relations

$$(3.1) \quad \dot{x}_i^{(k+1)} = f_i(x^{(k)}, t) + \sum_{j=1}^N (x_j^{(k+1)} - x_j^{(k)}) \frac{\partial f_i(x^{(k)}, t)}{\partial x_j},$$

$$i = 1, 2, \dots, N \quad \text{and} \quad k = 0, 1, 2, \dots,$$

with the boundary conditions

$$(3.2) \quad \sum_{j=1}^N a_j(t_i) x_j^{(k+1)}(t_i) = b_i, \quad i = 1, 2, \dots, N,$$

for  $k = 0, 1, 2, \dots$ . The linear system for  $x^{(k+1)}$ , equation (3.1), subject to the linear conditions of equation (3.2), is to be solved in the following manner. We determine a particular solution of equation (3.1) and  $N$  independent homogeneous solutions via numerical integration. This is readily accomplished since these are all initial value problems. Then  $x_i^{(k+1)}(t)$  is expressed as

$$(3.3) \quad x_i^{(k+1)}(t) = p_i^{(k+1)}(t) + \sum_{j=1}^N c_j h_{j,i}^{(k+1)}(t),$$

using an obvious notation. The  $N$  unknown constants,  $c_j$ , are then determined by solving the linear algebraic equations obtained by substituting the expression in (3.3) in equation (2.2).

In this way, the solution is effected by reducing the original problem to a sequence of linear initial value problems, a process well suited for machine calculation. Some aspects of convergence (which is quadratic in nature) for the case of two-point boundary value problems are discussed in [2]. We hope to discuss convergence questions in a future note.

#### 4. ORBIT DETERMINATION

Consider the problem of determining the unperturbed orbit of a heavenly body  $H$  in the  $(x,y)$  plane, the plane of the ecliptic, on the basis of four angular observations from the earth.

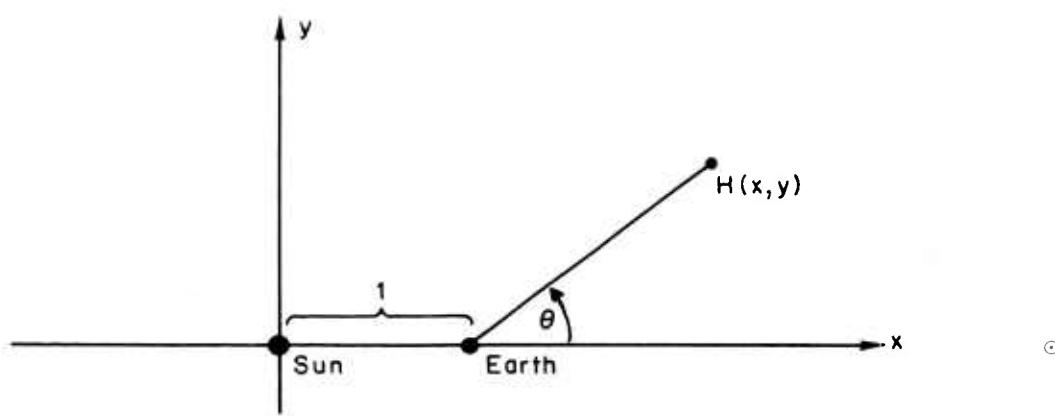


Fig. 1—The physical situation

Assume that the units are normalized so that the equations of motion of  $H$  are

$$(4.1) \quad \dot{x} = u, \quad \dot{u} = -x/(x^2 + y^2)^{1.5},$$

$$\dot{y} = v, \quad \dot{v} = -y(x^2 + y^2)^{1.5}.$$

At times  $t_i$ ,  $i = 1, 2, 3, 4$ , we are given that

$$(4.2) \quad \theta(t_i) = \theta_i,$$

which yields the conditions

$$(4.3) \quad y(t_i) = (x(t_i) - 1)\tan \theta_i, \quad i = 1, 2, 3, 4.$$

We wish to determine a complete set of elements of the orbit in the form of values for  $x$ ,  $u$ ,  $y$ , and  $v$  for some convenient time  $t$ . For simplicity, we are assuming that the earth is stationary and that the sole force on the heavenly body is the gravitational attraction of the sun.

## 5. NUMERICAL RESULTS

Consider the case for which the observational data are (in radians)

$$(5.1) \quad \theta(0.5) = 0.251297, \quad \theta(1.0) = 0.510240, \\ \theta(1.5) = 0.783690, \quad \theta(2.0) = 1.076540.$$

These data were generated by assuming that at time zero we have  $x(0) = 2.0$ ,  $y(0) = 0$ ,  $u(0) = 0$ ,  $v(0) = 0.5$ . The situation, with a stationary earth, is as shown in Fig. 2. As an

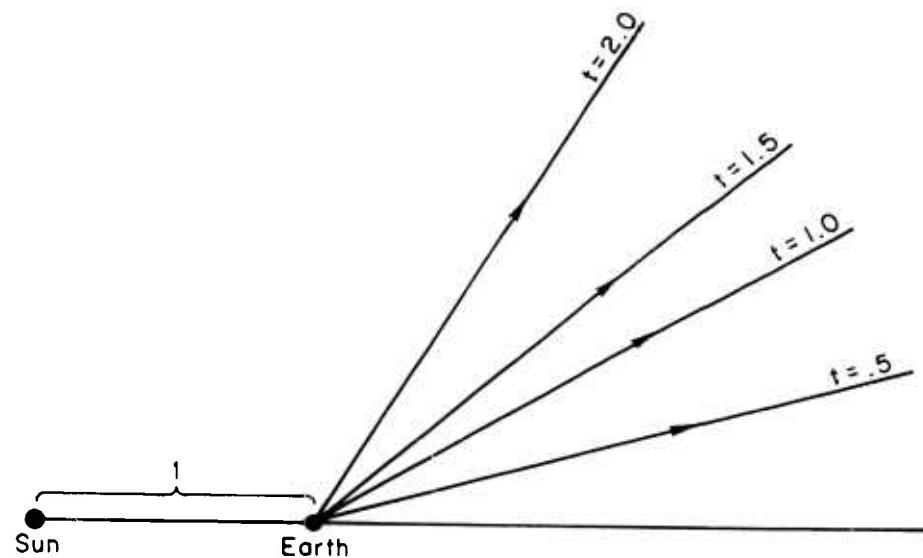


Fig. 2—Approximate directions to the heavenly body at four times of observation from a fixed earth

initial approximation to the trajectory, we assume that the observed body is at the point  $(1.0, 0)$  at time zero and that it moves in a circular orbit with a period equal to 3.664, a very poor initial approximation. Then using a Runge-Kutta integration method and the procedure sketched above, we found the following:

Table 1  
THE APPROXIMATIONS TO THE DISPLACEMENTS  
AND VELOCITIES AT TIME 2.5

	Initial Approx.	First Approx.	Second Approx.	Third Approx.	Fourth Approx.	Precise Value to 6 Figures
$x(2.5)$	-0.455175	1.554360	1.169660	1.190170	1.193560	1.193610
$u(2.5)$	-0.873320	-1.638830	-0.495101	-0.656686	-0.664207	-0.664263
$y(2.5)$	0.975790	2.982690	0.865184	1.043930	1.060540	1.060700
$v(2.5)$	-0.408690	1.807620	0.201206	0.237695	0.247433	0.247499

The calculation took about one and one-half minutes, no attempt having been made to streamline the calculations. It produced, of course, not merely the results of Table 1 but an ephemeris for  $0 \leq t \leq T$  at intervals of 0.01.

Perhaps even more surprising is the fact that if, as an initial approximation to the orbit, we consider the heavenly body to coincide with the position of the earth over all time, we still obtain the same rapid convergence to the correct orbit.

## 6. DISCUSSION

The method is readily extended to cases where more than four observations are available through use of the method of least squares. Furthermore, perturbations due to the motion of the earth and the presence of other planets are readily incorporated, since they merely result in modifications of equations (4.1). Of course, closely spaced observations present difficulties in the numerical determination of the constants  $c_j$  of equation (3.3). These matters and others will be discussed in more detail later.

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